Math exercises for revision: Solutions

Simplifications

Simplify the following expressions:

1.) $\frac{(ax)^2-a^2}{ax-a} = \frac{a^2(x^2-1)}{a(x-1)} = a\frac{(x+1)(x-1)}{x-1} = a(x+1)$

2.) $x(x-1)(x+3) - x^2(1+x) = x(x^2+3x-x-3) - x^2(1+x) =$
$x^3 + 2x^2 - 3x - x^2 - x^3 = x^2 - 3x = x(x-3)$

3.) $\ln(ax) + \ln(bx) - ln(c) = \ln\left(\frac{abx^2}{c}\right)$

4.) $\left(\frac{11x^{-3}}{4y^5}\right)^2 / \left(\frac{2}{y}\right)^{-3} = \left(\frac{121x^{-6}}{16y^{10}}\right) / \left(\frac{y^3}{8}\right) = \left(\frac{121x^{-6}}{16y^{10}}\right) \cdot \left(\frac{8}{y^3}\right) = \frac{121}{2x^6y^{13}}$

5.) $\sum_{i=1}^n 1 = n$

6.) $\prod_{i=1}^n 1 = 1$

Equations

Solve the following equations for x:

7.)

$$\begin{aligned}(x-7)(x+3) &= x(x+2)+5\\ x^2-4x-21 &= x^2+2x+5\\ -6x &= 26\\ x &= -\frac{13}{3} \approx -4.33\end{aligned}$$

8.)

$$\frac{3x^2}{x-1} - 3x = \frac{3}{x-1} + 2$$

Caution: Multiplying both sides by $x-1$ seems to yield $x=1$. However, for this x, the original equation is undefined (division by 0). Therefore, there is no solution.

9.)

$$\begin{aligned}4x + 2y - 6x^2 - 3xy &= (2 - 3x)(2x + y)\\ 4x + 2y - 6x^2 - 3xy &= 4x + 2y - 6x^2 - 3xy\end{aligned}$$

Both sides are identical, so the equation is true for all x.

10.)

$$2x^2 - 7x + 3 = 0$$

$$x = \frac{7 \pm \sqrt{49 - 24}}{4} = \frac{7 \pm \sqrt{25}}{4}$$

$$x \in \{\frac{1}{2}, 3\}$$

11.)

$$\begin{aligned}x^3 - yx^2 &= x - y\\ x^2(x - y) &= (x - y)\end{aligned}$$

This is true either if $x^2 = 1$, that is, $x = \pm 1$, or if $x - y = 0$, that is, $x = y$. Note that only in the first case could you divide by $x - y$.

12.)

$$\begin{aligned}\ln(a^x) &= 1/2\\ x\ln(a) &= 1/2\end{aligned}$$

$$x = \frac{1}{2\ln(a)} = \frac{1}{\ln(a^2)}$$

13.)

$$\begin{aligned}a^{rx} &= 100\\ \ln(a^{rx}) &= \ln(100)\\ rx\ln(a) &= \ln(100)\\ x &= \frac{\ln(100)}{r\ln(a)}\end{aligned}$$

Inequalities

Solve the following inequalities for x:

14.)

$$\begin{aligned} -\frac{x}{3} + 5 &> 15 \\ -\frac{x}{3} &> 10 \\ -x &> 30 \\ x &< 30 \end{aligned}$$

15.)

$$\begin{aligned} \frac{2-x}{3} &< \frac{4x-3}{2} \\ 2(2-x) &< 3(4x-3) \\ 4-2x &< 12x-9 \\ -14x &< -13 \\ x &> \frac{13}{14} \end{aligned}$$

16.)

$$\begin{aligned} \frac{x+3}{2} &< \frac{x+2}{-3} \\ -3(x+3) &> 2(x+2) \\ -3x-9 &> 2x+4 \\ -5x &> 13 \\ x &< -\frac{13}{5} \end{aligned}$$

Derivatives

Find the derivatives with respect to x:

17.) $\frac{\mathrm{d}}{\mathrm{d}x}(-x^2 + 8x - 12) = -2x + 8$

18.) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^2+1}\right) = -\frac{2x}{(x^2+1)^2}$ (Chain rule! Note that $(1/x)' = -1/x^2$)

19.) $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{3x} = 3\mathrm{e}^{3x}$

20.) $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^n + \mathrm{e}^{ax}\right) = nx^{n-1} + a\mathrm{e}^{ax}$

21.) $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^n \mathrm{e}^{ax}\right) = ax^n \mathrm{e}^{ax} + nx^{n-1}\mathrm{e}^{ax} = x^{n-1}\mathrm{e}^{ax}(ax+n)$

22.) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{ax^2}{1-x}\right) = \frac{(1-x)2ax+ax^2}{(1-x)^2} = \frac{2ax-ax^2}{(1-x)^2} = ax\frac{2-x}{(1-x)^2}$

23.) $\frac{\mathrm{d}}{\mathrm{d}x}\ln(x) = \frac{1}{x}$

24.) $\frac{\mathrm{d}}{\mathrm{d}x}\ln(ax^2) = \frac{2ax}{ax^2} = \frac{2}{x}$

Integrals

Determine the following integrals:

25.) $\int 3x^2 \mathrm{d}x = x^3 + C$, where C is the integration constant.

26.) $\int_{x=1}^{2} 8x^3 \mathrm{d}x = 2x^4\big|_1^2 = 32 - 2 = 30$

27.) $\int_{x=0}^{3}(2x+1)\mathrm{d}x = (x^2+x)\big|_0^3 = 12$

28.) $\int_{x=0}^{\infty} \mathrm{e}^{-rx}\mathrm{d}x = -\frac{e^{-rx}}{r}\Big|_0^{\infty} = 0 + 1/r = 1/r$

Limits

Determine the following limits:

29.) $\lim_{x\to\infty} \frac{\mathrm{e}^x}{x^{39874}} = \infty$ (If you don't believe this, apply L'Hopital's rule 39874 times. e^x increases faster than x to any power you can imagine.)

30.) $\lim_{x\to\infty} \frac{3x^2+1}{2x^2-5} = \frac{3}{2}$ (For large x, the constants 1 and -5 are negligible.)

31.) $\lim_{x\to 1} \frac{x^4-4}{x^3-1}$. For $x = 1$, the numerator is -4 and the denominator is 0. Thus, we have to distinguish whether x approaches 1 from above or from below. We get $\lim_{x\to 1\downarrow} \frac{x^4-4}{x^3-1} = -\infty$ and $\lim_{x\to 1\uparrow} \frac{x^4-4}{x^3-1} = \infty$. Actually, I commited a typo here. What I really wanted you to do is to calculate $\lim_{x\to 1} \frac{x^4-1}{x^3-1}$. Then, plugging in $x = 1$ yields $0/0$, so we have to use L'Hopital's rule to get $\lim_{x\to 1} \frac{x^4-1}{x^3-1} = \lim_{x\to 1} \frac{4x^3}{3x^2} = 4/3$.