Exercise 5

Part 1: Metapopulation dynamics

One-dimensional models are not only used to study changes in the size of a single population. By reducing the local population dynamics to two states (absent or present), we can also use a one-dimensional model to study so-called metapopulations. A metapopulation is a population of populations of the same species, with each local population living in one of many patches in a landscape. Local populations can go extinct, for example because of a fire, a disease etc. (analogous to death events), and they can be recolonized by migrants from other local populations (analogous to birth events). Infection-recovery dynamics can also be understood as metapopulation dynamics: the host individuals are the patches, and transmission and recovery correspond to recolonization and extinction, respectively.

A simple discrete-time model for metapopulation dynamics is:

$$p_{t+1} = (1-e)(p_t + mp_t(1-p_t)) =: f(p_t), \tag{1}$$

where p_t is the proportion of patches that are occupied at time t, e (with $0 \leq e \leq 1$) is the probability that a local population goes extinct from one time step to the next, and $m > 0$ determines how fast empty patches are recolonized by members of occupied patches.

1.1 For what combinations of e and m do we obtain a biologically meaningful model? What is the problem if m is too large? How could we modify the model to solve this issue?

1.2 Compute the equilibria of model (1). For those regions of parameter space where the equilibria are biologically meaningful, determine their stability and the behavior around the equilibria (no need to investigate limit cycles or chaos in detail). Sketch a cobwebbing plot for each qualitatively different case.

1.3 With your results from the last two exercises, draw a picture of the (e, m)-parameter space and divide it into regions with qualitatively different behavior. Explain in biological terms what happens in each region.

Part 2: Global stability of the Lotka-Volterra Model

The two-species Lotka-Volterra model is defined by the equations

$$\dot{x} = f_x(x,y) = r_x x + c_x x^2 + c_{xy} x\, y\,, \tag{2a}$$

$$\dot{y} = f_y(x,y) = r_y y + c_y y^2 + c_{yx} y\, x\,. \tag{2b}$$

2.1 Consider the competition model (2) with $r_x, r_y > 0$ and $c_x, c_y, c_{xy}, c_{yx} < 0$. Show that the quadratic form

$$Q(x,y) = c_x c_{yx}(x - x_4^*)^2 + 2c_{xy}c_{yx}(x - x_4^*)(y - y_4^*) + c_{xy}c_y(y - y_4^*)^2$$

is a Lyapunov function for this model.

2.2 Use this function and Lyapunov's theorem to exclude periodic orbits and to determine all parameter regions of global stability of the competition model.

2.3 We can use Lyapunov functions also to prove global stability for a periodic attractor. Consider the dynamical system

$$\dot{x} = f_x(x, y) = x - y - x(x^2 + y^2)\,, \tag{3a}$$
$$\dot{y} = f_y(x, y) = x + y - y(x^2 + y^2)\,. \tag{3b}$$

Use Lyapunov's theorem and the function $V(x, y) = (1 - x^2 - y^2)^2$ to prove that the unit circle is a global periodic attractor.