Exercise 5: Dynamics of interacting populations

The two-species Lotka-Volterra model is defined by the equations

$$\dot{x} = f_x(x, y) = r_x x + c_x x^2 + c_{xy} x\, y\,, \tag{1a}$$

$$\dot{y} = f_y(x, y) = r_y y + c_y y^2 + c_{yx} y\, x\,. \tag{1b}$$

Part 1: Mutualism

Consider (1) with mutualism ($c_{xy}, c_{yx} > 0$) and intraspecific competition ($c_x, c_y < 0$). One distinguishes two basic types of mutualism:

- With *obligate mutualism*, neither species can survive without the presence of the other species. We then have $r_x, r_y < 0$.
- With *facultative mutualism*, the other species is helpful, but not essential. This is characterized by $r_x, r_y > 0$.

4.1 First consider the case of obligate mutualism. Draw possible isocline configurations, add flow arrows, determine the equilbria and perform a stability analysis. You should find two qualitatively different parameter regimes. Can you exclude periodic orbits? What is the biological interpretation of the different regimes (examples)?

4.2 Do the same for the case of facultative mutualism. Discuss under what conditions this model might or might not make reasonable predictions. How could we make the model more realistic (by going beyond the Lotka-Volterra scheme)?

Part 2: Global stability

4.3 Consider the competition model (1) with $r_x, r_y > 0$ and $c_x, c_y, c_{xy}, c_{yx} < 0$. Show that the quadratic form

$$Q(x, y) = c_x c_{yx}(x - x_4^*)^2 + 2c_{xy}c_{yx}(x - x_4^*)(y - y_4^*) + c_{xy}c_y(y - y_4^*)^2$$

is a Lyapunov function for this model.

4.4 Use this function and Lyapunov's theorem to exclude periodic orbits and to determine all parameter regions of global stability of the competition model.

4.5 We can use Lyapunov functions also to prove global stability for a periodic attractor. Consider the dynamical system

$$\dot{x} = f_x(x, y) = x - y - x(x^2 + y^2)\,, \tag{2a}$$

$$\dot{y} = f_y(x, y) = x + y - y(x^2 + y^2)\,. \tag{2b}$$

Use Lyapunov's theorem and the function $V(x, y) = (1 - x^2 - y^2)^2$ to prove that the unit circle is a global periodic attractor.