Exercise 6:

Part 1: SIR Model

In this exercise, we will study the epidemiology of an infectious disease, e.g. a virus or a bacterial infection. We will keep track of the densities of susceptible host individuals, S, infected host individuals, I, and recovered host individuals, R. Consider the following model:

$$\dot{S} = (b - c(S + I + R))(S + I + R) - dS - \beta SI, \tag{1}$$

$$\dot{I} = \beta SI - (\alpha + d + \nu)I, \tag{2}$$

and

$$\dot{R} = \nu I - dR, \tag{3}$$

where $b, c, d, \alpha, \beta, \nu > 0$, and $b > d$.

4.1 Give names to all the model parameters and describe the biological assumptions underlying the model. Describe the host dynamics in the absence of disease. What is the equilibrium? Derive the stability of this equilibrium in the presence of the disease.

4.2 Arguably the most important quantity in epidemiology is the *basic reproductive ratio* R_0, defined as the average number of new infections resulting from a single infected individual introduced into a completely susceptible population. What is R_0 in our model? Formally show that the disease can spread if and only if $R_0 > 1$. Discuss the concept of a *critical community size*, K_0, of the host for the spread of a disease.

4.3 Assume now that a proportion p of newborns is vaccinated and becomes resistant for life. Write down the differential equations for this modified model. What proportion of newborns needs to be vaccinated to eradicate the disease? (Hint: derive the new R_0.) In this context, discuss the meaning of *herd immunity*.

4.4 The above model assumes *density-dependent transmission*, i.e. the rate at which susceptible individuals get infected is proportional to the density of infected individuals, I. For what kinds of infectious disease do you think this is a realistic assumption, thinking about i) flu, ii) vector-borne diseases, iii) sexually transmitted diseases? How could we modify the model to make it more realistic for sexually transmitted diseases? How does this change the criterion for disease spread? What are the implications for the critical community size?

Part 2: Coexistence in simple ecosystems

Consider a general two-species competition model

$$\dot{x}_1 = x_1(r_1 - c_{11}x_1 - c_{12}x_2) \tag{4}$$

$$\dot{x}_2 = x_2(r_2 - c_{21}x_1 - c_{22}x_2) \tag{5}$$

with $r_i, c_{ij} > 0$. For simplicity, we can fix $r_1 = r_2 = 1$.

4.5 Assume that one of the species (say species 2) dominates the other species. Define a suitable third competitor such that the three-species system (with $r_i = 1$, $c_{ij} > 0$) allows stable coexistence. What are appropriate values for the c_{3i} and c_{i3}? For the purpose of this exercise it is sufficient to show that all boundary equilibria are unstable ("invasibility"). Does this imply that an internal stable equilibrium exists?

4.6 Show that it is not possible to stabilize the system in this way if species 1 and 2 are mutually exclusive.

4.7 Try the same as in (4.5) and (4.6) with introduction of a predator as third species ($\dot{x}_3 = x_3(-1 + c_{31}x_1 + c_{32}x_2 - c_{33}x_3)$ with $c_{3i} \geq 0$) instead of a competitor.

Exercises (4.6) and (4.7) are for extra points.