Dynamics of heterozygosity under assortative mating

We are considering the assortative mating case discussed in exercise 1.2 and 1.3. The quantity $H = 2P_{12}$ is known as *heterozygosity.* ($2P_{12}$ is the frequency of all heterozygotes, such that $P_{11} + 2P_{12} + P_{22} = 1$.)

1. Derive the recursion for H' (which can be expressed as a function of H and ρ).
2. For the case $\rho = 1$, derive the value of $H(t)$ in generation t given e.g. $p(t = 0) = \frac{1}{2}$.
3. What general conclusion can be derived for $\lim_{t\to\infty} H(t)$ in the case of $\rho < 1$?

Solution:

1. In the previous exercise, we found the following recursion:

$$2P'_{12} = 2(1-\rho)pq + 2\rho p\left(1 - \frac{p}{1-P_{22}}\right),$$

Writing $p = P_{11} + P_{12}$ and keeping in mind that $P_{11} + 2P_{12} + P_{22} = 1$, we obtain $1 - P_{22} = P_{11} + 2P_{12} = p + P_{12} = p + \frac{1}{2}H$, and thus

$$\begin{aligned} H' &= 2(1-\rho)pq + 2\rho p\left(1 - \frac{p}{p+\frac{1}{2}H}\right) \\ &= 2(1-\rho)pq + 2\rho p\frac{\frac{1}{2}H}{p+\frac{1}{2}H} \end{aligned}$$

2. In the special case when $\rho = 1$ (complete assortative mating), the previous recursion becomes:

$$H' = \frac{pH}{p+\frac{1}{2}H}.$$

We can study the quantity $\frac{1}{H}$. The recursion becomes:

$$\begin{aligned} \left(\frac{1}{H}\right)' &= \frac{p+\frac{1}{2}H}{pH} \\ &= \frac{1}{H} + \frac{1}{2p} \\ \Rightarrow \frac{1}{H(t)} &= \frac{1}{H(0)} + \frac{t}{2p} \\ \Rightarrow H(t) &= \frac{1}{\frac{1}{H(0)} + \frac{t}{2p}} \\ &= \frac{2p}{\frac{2p}{H(0)} + t} \end{aligned}$$

Finally for $p = \frac{1}{2}$ we obtain

$$H(t) = \frac{1}{\frac{1}{H(0)} + t} \xrightarrow{t\to\infty} 0$$

3. In the general case, the heterozygosity may evolve towards an equilibrium value H^* satisfying the following equation:

$$\begin{aligned}
H^* &= 2(1-\rho)pq + \frac{\rho p H^*}{p + \frac{1}{2}H^*} \\
\Leftrightarrow \left(p + \frac{1}{2}H^*\right) H^* &= 2(1-\rho)pq\left(p + \frac{1}{2}H^*\right) + \rho p H^* \\
&\Leftrightarrow H^{*2} + H^*\left(2p - 2(1-\rho)pq - 2\rho p\right) - 4(1-\rho)p^2 q = 0 \\
&\Leftrightarrow H^{*2} + 2p^2(1-\rho)H^* - 4(1-\rho)p^2 q = 0
\end{aligned}$$

The roots can be found the usual way:

$$\Delta = 4p^4(1-\rho)^2 + 16(1-\rho)p^2 q = 4p^4(1-\rho)^2\left(1 + \frac{4q}{p^2(1-\rho)}\right) > 0$$

Only one root is positive:

$$\begin{aligned}
H^* &= \frac{-2p^2(1-\rho) + \sqrt{\Delta}}{2} \\
&= p^2(1-\rho)\left(\sqrt{1 + \frac{4q}{p^2(1-\rho)}} - 1\right)
\end{aligned}$$